Sem I # 1.4 (Examples) - Hyperbolic functions

Ex (1) Separate the following expression into their real and imaginary parts (i) $\sin(x+iy)$ (ii) ~~cos~~ $\sinh(x+iy)$

Solution:- (i) $\sin(x+iy) = \sin x\cdot\cos iy + \cos x\cdot\sin iy$

$$= \sin x\cdot\cosh y + \cos x\cdot i\sinh y$$

$$= \cos x\cdot\cosh y + i(\cos x\cdot\sinh y)$$

$$= A + iB \text{ form}$$

(ii) $\sinh(x+iy) = -i\sin i(x+iy)$ $\{\sinh x = -i\sin ix\}$

$$= -i\sin(ix - y)$$

$$= -i(\sin ix\cdot\cos y - \cos ix\cdot\sin y)$$

$$= -i(i\sinh x\cdot\cos y - \cosh x\cdot\sin y)$$

$$= -i^2\sinh x\cdot\cos y + i\cosh x\cdot\sin y$$

$$= \sinh x\cdot\cos y + i\cosh x\cdot\sin y$$

Ans

Ex (2) Separate $\log\sin(\theta+i\phi)$ into real and Imaginary parts.

Sol^n:- $\sin(\theta+i\phi) = \sin\theta\cdot\cos i\phi + \cos\theta\cdot\sin i\phi$

$$= \sin\theta\cdot\cosh\phi + \cos\theta\cdot i\sinh\phi$$

$$= \alpha + i\beta \quad \text{where } \alpha = \sin\theta\cdot\cosh\phi$$
$$\beta = \cos\theta\cdot\sinh\phi$$

Now, $\log\sin(\theta+i\phi) = \log(\alpha+i\beta)$

$$= \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\frac{\beta}{\alpha}$$

$$= \frac{1}{2}\log(\sin^2\theta\cdot\cosh^2\phi + \cos^2\theta\cdot\sinh^2\phi) + i\tan^{-1}\frac{\cos\theta\cdot\sinh\phi}{\sin\theta\cdot\cosh\phi}$$

$$= \frac{1}{2}\log[\sin^2\theta(1+\sinh^2\phi) + (1-\sin^2\theta)\sinh^2\phi] + i\tan^{-1}(\cot\theta\ \cancel{\sinh\phi})$$

$$= \frac{1}{2}\log[\sin^2\theta + \cancel{\sin^2\theta\cdot\sinh^2\phi} + \sinh^2\phi - \cancel{\sin^2\theta\sinh^2\phi}] + i\tan^{-1}(\cot\theta\cdot\tanh\phi)$$

$$= \frac{1}{2}\log(\sin^2\theta + \sinh^2\phi) + i\tan^{-1}(\cot\theta\ \tanh\phi)$$

Ans

Ex ③ If $\sin(\theta + i\phi) = x + iy$, Prove that

(i) $\dfrac{x^2}{\sin^2\theta} - \dfrac{y^2}{\cos^2\theta} = 1$ (ii) $\dfrac{x^2}{\cosh^2\phi} + \dfrac{y^2}{\sinh^2\phi} = 1$

Solution:- Given that

$\sin(\theta + i\phi) = x + iy$

$\Rightarrow \sin\theta \cdot \cos i\phi + \cos\theta \cdot \sin i\phi = x + iy$

$\Rightarrow \sin\theta \cdot \cosh\phi + \cos\theta \cdot i\sinh\phi = x + iy$

$\therefore \quad x = \sin\theta \cdot \cosh\phi \quad \text{—(i)}$

$y = \cos\theta \cdot \sinh\phi \quad \text{—(ii)}$

(i) LHS $= \dfrac{x^2}{\sin^2\theta} - \dfrac{y^2}{\cos^2\theta}$

$= \dfrac{\cancel{\sin^2\theta} \cdot \cosh^2\phi}{\cancel{\sin^2\theta}} - \dfrac{\cancel{\cos^2\theta} \cdot \sinh^2\phi}{\cancel{\cos^2\theta}}$

$= \cosh^2\phi - \sinh^2\phi = 1 = \text{RHS}$

(ii) LHS $= \dfrac{x^2}{\cosh^2\phi} + \dfrac{y^2}{\sinh^2\phi}$

$= \dfrac{(\sin\theta \cdot \cosh\phi)^2}{\cosh^2\phi} + \dfrac{(\cos\theta \cdot \sinh\phi)^2}{\sinh^2\phi}$

$= \sin^2\theta + \cos^2\theta = 1 = \text{RHS}$

Ex ④ If $\sin(\theta + i\phi) = u + iv$, Prove that $\sin^2\theta$ and $\cosh^2\phi$ are the roots of eqn

$t^2 - t(1 + u^2 + v^2) + u^2 = 0$

Solution:- Given that

$\sin(\theta + i\phi) = u + iv$

$\sin\theta \cdot \cos i\phi + \cos\theta \cdot \sin i\phi = u + iv$

$\sin\theta \cdot \cosh\phi + \cos\theta \cdot i\sinh\phi = u + iv$

Comparing real & Imaginary parts,

$$u = \sin\theta \cdot \cosh\phi \quad \text{—(i)}$$
$$v = \cos\theta \cdot \sinh\phi \quad \text{—(ii)}$$

Now,

$$u^2 + v^2 = \sin^2\theta \cdot \cosh^2\phi + \cos^2\theta \cdot \sinh^2\phi$$

$$= \sin^2\theta \cdot \cosh^2\phi + (1 - \sin^2\theta)\cdot(\cosh^2\phi - 1)$$

$$= \cancel{\sin^2\theta \cdot \cosh^2\phi} + \cosh^2\phi - 1 - \cancel{\sin^2\theta \cdot \cosh^2\phi} + \sin^2\theta$$

$$= \cancel{\cosh^2\phi - (1 - \sin^2\theta)}$$

$$= \sin^2\theta + \cosh^2\phi - 1 \quad \text{—(iii)}$$

Now, Sum of roots $= \sin^2\theta + \cosh^2\phi = 1 + u^2 + v^2$ {(iii)}

Product of roots $= \sin^2\theta \cdot \cosh^2\phi = u^2$ {from (i)}

The quadratic eq^n whose roots $\sin^2\theta$ and $\cosh^2\phi$

$$t^2 - (\text{Sum of roots})t + \text{Product of roots} = 0$$

$$t^2 - (1 + u^2 + v^2)t + u^2 = 0$$

Proved

Ex(5) If $u = \log_e \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$, prove ~~that~~

(i) $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$

(ii) $\cos\theta \cdot \cosh u = 1$

(iii) $\sinh u = \tan\theta$

(iv) $\tanh u = \sin\theta$

(v) If $u = \theta + a_3\theta^3 + a_5\cdot\theta^5 + \dots$ Prove that

$$\theta = u - a_3 u^3 + a_5 u^5 - \dots$$

Proof:- (i) Given $u = \log_e \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

$$e^u = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{\tan\frac{\pi}{4} + \tan\theta/2}{1 - \tan\pi/4 \cdot \tan\theta/2}$$

$$\frac{e^u}{1} = \frac{1 + \tan\theta/2}{1 - \tan\theta/2} \quad \text{—(i)}$$

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$$\frac{e^u+1}{e^u-1} = \frac{(1+\tan\theta/2)+(1-\tan\theta/2)}{(1+\tan\theta/2)-(1-\tan\theta/2)}$$

$$\Rightarrow \frac{\cancel{e^{u/2}}\left(e^{u/2}+\frac{1}{e^{u/2}}\right)}{\cancel{e^{u/2}}\left(e^{u/2}-\frac{1}{e^{u/2}}\right)} = \frac{\cancel{2}}{\cancel{2}\tan\theta/2}$$

$$\Rightarrow \frac{\left(e^{u/2}+e^{-u/2}\right)/2}{\left(e^{u/2}-e^{-u/2}\right)/2} = \frac{1}{\tan\theta/2}$$

$$\Rightarrow \frac{\cosh\frac{u}{2}}{\sinh\frac{u}{2}} = \frac{1}{\tan\theta/2} \Rightarrow \boxed{\tan\frac{\theta}{2} = \tanh\frac{u}{2}}$$ Proved (i)

(ii) from (i) $\frac{1}{e^u} = \frac{1-\tan\theta/2}{1+\tan\theta/2} \Rightarrow e^u = \frac{1+\tan\theta/2}{1-\tan\theta/2}$ —— (ii)

and $e^{-u} = \frac{1-\tan\theta/2}{1+\tan\theta/2}$ —— (iii)

Adding (ii) & (iii)

$$e^u+e^{-u} = \frac{1+\tan\theta/2}{1-\tan\theta/2} + \frac{1-\tan\theta/2}{1+\tan\theta/2}$$

$$= \frac{1+\cancel{2\tan\theta/2}+\tan^2\theta/2+1-\cancel{2\tan\theta/2}+\tan^2\theta/2}{1-\tan^2\theta/2}$$

$$e^u+e^{-u} = \frac{2(1+\tan^2\theta/2)}{1-\tan^2\theta/2}$$

$$\Rightarrow \frac{e^u+e^{-u}}{2} = \frac{1}{\cos\theta} \Rightarrow \cosh u = \frac{1}{\cos\theta}$$ —— (iv)

$$\Rightarrow \boxed{\cosh u \cdot \cos\theta = 1}$$ Proved

(iii) $\cosh u \cdot \cos\theta = 1 \Rightarrow \cosh u = \frac{1}{\cos\theta} = \sec\theta$

Squaring, $\cosh^2 u = \sec^2\theta \Rightarrow \cosh^2 u - 1 = \sec^2\theta - 1$

$$\Rightarrow \sinh^2 u = \tan^2\theta \Rightarrow \boxed{\sinh u = \tan\theta}$$ (v)